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Complexity of critical functions for Hamiltonian systems

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Abstract. Transition to predominantly chaotic motion in Hamiltonian systems with two degrees of freedom is described by a complicated function of frequency, which is called the critical function. A graph of this function is a fractal set with the local structure, which is believed to depend only on the arithmetic nature of the frequency. We calculated numerically fractal dimensions of this function for a few typical systems using the method of modular smoothing and an efficient algorithm for computation of the fractal dimensions. The dimensions which measure the complexity of the fractal are indeed the same, within the error bounds, and are equal to the dimension of the exponent of the Brjuno function, which is a purely arithmetic function.

1. Introduction

The phase space of a typical nonlinear Hamiltonian system with two degrees of freedom consists of a mixture of regular and chaotic orbits with a complicated fractal boundary between them. The regular orbits are restricted on KAM tori and are quasi-periodic while the chaotic orbits are unrestricted with a broad frequency spectrum. As the nonlinearity is increased many of the KAM tori disappear, and a measure of the phase space with predominantly irregular orbits increases. Typical features of the transition to predominantly irregular motion are usually studied using one parameter families of area-preserving maps or non-autonomous Hamiltonian system with one and a half degrees of freedom and continuous time [1].

Typical and most studied examples are the Taylor–Chirikov standard map (SM), given by the following equations:

$$y' = y + k\sin(x) \qquad x' = x + y' \tag{1}$$

and the two-waves model of Zaslavsky, Escande and Doveil given by the following Hamiltonian:

$$H = p^2/2 - k(\exp iq + \exp i(q - t)). \tag{2}$$

It is generally believed that the typical properties of these systems are also present in the corresponding complex area-preserving maps and systems with a complex Hamiltonian [2–4]. Such systems are generally much easier to analyse both algorithmically and analytically than the real Hamiltonian systems. An example of such systems which we shall use in this paper is the semi-standard map [2] given by

$$y' = y + \tfrac{1}{2}ik\exp(ix) \qquad x' = x + y'. \tag{3}$$

The KAM tori of these systems are parametrized by an irrational frequency ν of the quasi-periodic trajectories on the tori. At zero value of the parameter k the system is

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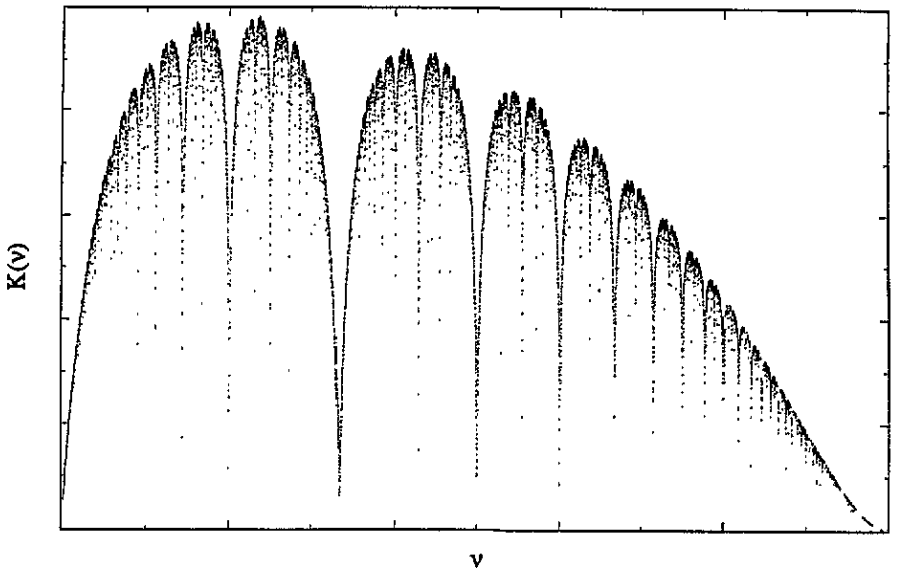

Figure 1. The figure presents the critical function of the semi-standard map.

integrable and the phase space is filled by invariant tori. For a sufficiently large value of the parameter the KAM torus with a frequency ν is no longer a smooth function of the unperturbed torus [5]. The smallest value of the perturbation parameter k at which the KAM torus with the frequency ν is destroyed is called the critical perturbation. The dependence of the critical perturbation on the frequency is called the critical function, and is denoted by $K(\nu)$ [2]. Let us remark that an analogous critical function can be defined in the case of a more general Hamiltonian system with two degrees of freedom which satisfy conditions of KAM theory. In this case we consider a one parameter family of non-autonomous systems defined on hyperplanes of constant energy.

The critical function was introduced in [2] and since then has been intensely studied [6]. Properties of this complicated fractal function are described in detail in [3]. It is believed that the local structure of the critical functions depends essentially only on the arithmetic properties of the frequency, and not on the dynamics. The aim of this paper is to present the calculations of fractal dimensions of the critical functions for a few typical Hamiltonian systems, and to show that these dimensions are indeed independent of the dynamics. Thus, the geometric complexity of the critical functions for a class of Hamiltonian systems does not depend on the dynamics.

2. Critical functions and the method of modular smoothing

General properties of the critical functions follow from KAM theory [10]. A typical example of such functions is given in figure 1. It is zero and continuous at all rational values of the frequency, and non-zero and discontinuous at irrational frequencies satisfying some Diophantine condition. A necessary condition on the frequency such that $K(\nu)$ is non-zero is still not known. There is clear evidence of a self-similar structure of the critical function. Local maxima are at noble numbers, that is numbers which are related to a golden mean γ by a finite number of Gauss transformations $\nu \to \nu' = 1/\nu - [1/\nu]$ where

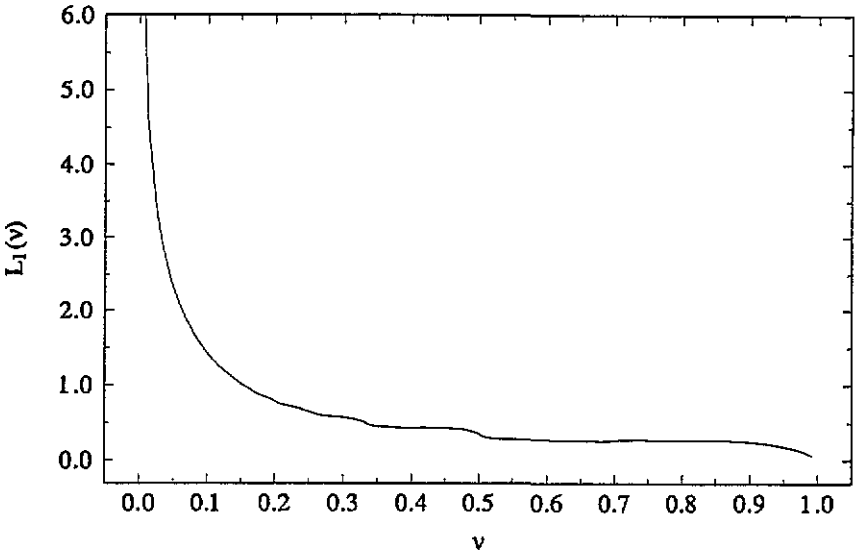

Figure 2. The function $L_1(\nu)$ for the semi-standard map.

$[1/\nu]$ is the integer part of $1/\nu$. Properties of the critical function are a consequence of the famous small divisors that appear in the perturbation expansion of the KAM tori, and other related problems, as in the problem of Siegel disks for analytic maps of a complex plain [7, 4] and in the context of diffeomorphisms of a circle [8, 9]. Detailed analyses of a perturbation expansion of the KAM torus with frequency ν reveals that the singularities at rational values of the frequency are all of the same functional form. The fact that the singularities are of the same form enables one to express transformation properties of the critical function under the action of an inversion and a unit translation on the frequency, by functions of the frequency which can be easily interpolated by smooth functions. This was used in [3] to introduce the method of modular smoothing, which provides us with an efficient and accurate approximate computation of the critical function which involves only the perturbation expansion to some finite and low order. (For later developments of the method see also [11] and [12]). Direct computation of the critical function, using the Greene method [13] or as a radius of convergence of the KAM torus [2, 3], involves estimating limits of infinite sequencies of periodic orbits or perturbation coefficients for each frequency.

Here we use the method of modular smoothing for calculation of the approximate $K(\nu)$ at sufficiently many frequencies, which are needed for reliable estimates of the fractal dimensions. Let us briefly summarize the main steps in the method of modular smoothing.

(i) A function $L_1(\nu) = \ln K(\nu) - \nu \ln K(\nu')$ is a continuous function of the frequency. Here, as always in this paper, ν' denotes $\nu' = 1/\nu - [1/\nu]$. An example is given in figure 2.

(ii) Values of the function L_1 at rationals $m/n \in [0, 1]$ are given by the following formula:

$$L_1(m/n) = 1/n \lim_{\nu \to m/n} \ln \frac{nb(n, m; \nu)}{mb(n', m'; \nu)} \tag{4}$$

where $b(n, m; \nu)$ are essentially coefficients of order n in a suitable perturbation expansion, and $n' = m$ and $m' = n - m[n/m]$. Details are given in [11, 12].

If the frequency ν is a noble number, d_ν steps away from the golden mean frequency γ,

the relation in (i) can be iterated $d_\nu - 1$ times until $\nu' = \gamma$. In this way we obtain

$$K(\nu) = K(\gamma)^{(\nu'\nu''\ldots\nu^{d_\nu-1})} \exp - \left[L_1(\nu) + L_1(\nu')\nu + L_1(\nu'')\nu\nu' \ldots + L_1(\nu^{d_\nu-1})\nu^{d_\nu-2}\ldots\nu\right]. \tag{5}$$

We have thus expressed the fractal function $K(\nu)$ using the continuous function $L_1(\nu)$. A good approximation of the critical function can now be obtained by calculating the function $L_1(m/n)$ at only a few m/n with small n, and interpolating the function $L_1(\nu)$ using a piecewise linear approximation through the points $L_1(m/n)$. This can be used to obtain an approximate critical function at any noble frequency. For other frequencies we first approximate the frequency by a noble approximation, which introduces an arbitrary small error in the calculation of the critical function. The error in the approximate critical function is not more than a couple of per cent, and the structure of the singularities is well approximated.

An analogous critical function is defined in the context of Siegel disks and circle diffeomorphisms [4]. Yoccoz has shown that the ratios of the logarithm of the critical functions for these two problems and a Brjuno function (defined below) [14], which is a purely arithmetic function, are bounded [15]. Later, Marmi [4] proved for the case of the semi-standard map that, provided the perturbation is small enough, the convergence of the Brjuno function at a frequency ν is a necessary and sufficient condition for the existence of the KAM torus with this frequency. He also conjectured that the ratio of the logarithms of the critical functions for the semi-standard map and the standard map, and the Brjuno function are bounded.

The Brjuno function $B(\nu)$ is related to a modified continued fraction expansion, and can be uniquely defined in terms of its transformation properties with respect to the unit translations and inversions [4]. It satisfies the following two relations:

$$B(\nu + 1) = B(-\nu) = B(\nu) \tag{6}$$

and for irrational $\nu \in \left(0, \frac{1}{2}\right)$

$$B(\nu) - \nu B(\nu^{-1}) = -\ln B(\nu). \tag{7}$$

We present numerical evidence that the fractal dimensions of the exponent of the Brjuno function are the same within the level of accuracy as the dimensions of the critical functions treated here. This again indicates that the complexity of the critical functions for a class of dynamical problems is of number-theoretic nature and is independent of the dynamics.

3. Fractal dimensions of the critical functions

In order to measure the complexity of the critical functions we use the Renyi spectre of fractal dimensions [16]. The dimensions can be defined as follows. We cover the graph of the critical function by $N(\epsilon)$ volume elements of diameter ϵ, and for every $q = 0, 1, 2\ldots$ we define the generalized entropy by the following formula:

$$H(\epsilon) = \frac{1}{1-q} \ln \sum_{i=1}^{N(\epsilon)} P_i^q(\epsilon) \qquad q \neq 1 \tag{8}$$

and

$$H(\epsilon) = \sum_{i=1}^{N(\epsilon)} P_i^q(\epsilon) \ln P_i^q(\epsilon) \qquad q = 1 \tag{9}$$

where $P_i(\epsilon)$ is the number of points on the graph of the critical function which is in the ith volume element. The Renyi dimensions are defined as follows:

$$D(q) = \lim_{\epsilon \to 0} \frac{H_q(\epsilon)}{\ln \epsilon}. \tag{10}$$

For $q = 0, 1, 2$ one recovers the most common fractal dimensions, namely the capacity, the information and the correlation dimensions, respectively. Generalizations to negative and arbitrary real dimensions have been introduced by Badii and Politi [17], and studied intensely as a way of characterizing chaotic attractors in dissipative systems [18] and other fractals [19]. However, we restrict our attention to integer positive dimensions only. Standard algorithms for calculating the dimensions numerically can be found, for example, in [20, 19].

One is faced with two main problems in trying to estimate numerically the fractal dimensions of realistic fractals. Firstly one needs quite a large number of points on the fractal so that the local structure becomes apparent, and secondly a straightforward algorithm based on the definition is quite memory consuming since one needs a large matrix to enumerate the volume elements in a sufficiently fine covering. We overcome these difficulties by using the method of modular smoothing to calculate the critical functions, and by using a simple trick in order to label and count the elements of the matrix using a one-dimensional array. Calculation of the dimensions for the critical functions poses an additional difficulty. Namely, numerical calculation of the dimensions are done using large but finite subsets of the graph of $K(\nu)$, where νs are such that $K(\nu)$ is most easily calculated. Such νs are the noble numbers, that is the most irrational numbers in the sense of their continued fraction approximation. In order to estimate the dimensions correctly we need to use such numbers to approximate the critical function at sufficiently many randomly distributed numbers.

Let us first consider the critical function for the SSM. We use this model to show that the fractal dimensions of an approximate critical function obtained by the method of modular smoothing and the exact critical function are the same. In this case by the exact critical function we mean the radius of convergence of the KAM tori as a function of the frequency [2]. Namely, a Fourier–Taylor expansion in the variable θ and the parameter k of the KAM torus with the frequency ν of the SSM can be replaced by a single Taylor expansion in a new variable $u = k \exp(-i\theta)$. The radius of convergence of this Taylor expansion turns out to be equal to the critical perturbation for this frequency, that is to the critical function. We call this radius of convergence the exact critical function. However, there are practical difficulties in applying this definition for calculating the critical function for frequencies that are quite close to a rational. For example, consider two irrationals with the same initial finite string in their continued fraction expansion. If this initial string is long it is practically impossible to distinguish the two irrationals although one of them could be quite close to a rational due to a large coefficient in the non-equal part of the continued fraction expansion. Consequently the critical function at the two irrationals could take quite different values due to differences in the continued fraction expansion which cannot be estimated using the perturbation expansion of a finite order. If the critical function is to be calculated with the large set of points needed for the calculation of the fractal dimensions, one needs to take special care to include correctly the effect of the points which are very close to the rationals, and for which the critical function cannot be calculated with sufficient accuracy.

We have calculated the fractal dimensions of two sets $(\nu_i, K(\nu_i))$ and $(\nu_i, K_a(\nu_i))$, where $\nu_i = i \times \gamma - [i \times \gamma]$, and $K(\nu)$ and $K_a(\nu)$ are the 'exact' critical function and its approximation by the method of modular smoothing. The results are presented in figure 3. Here $i = 1, 2 \ldots 20\,000$, but points too close to rationals had to be neglected since, for these points, we could not calculate reliably the 'exact' critical function. The approximation K_a

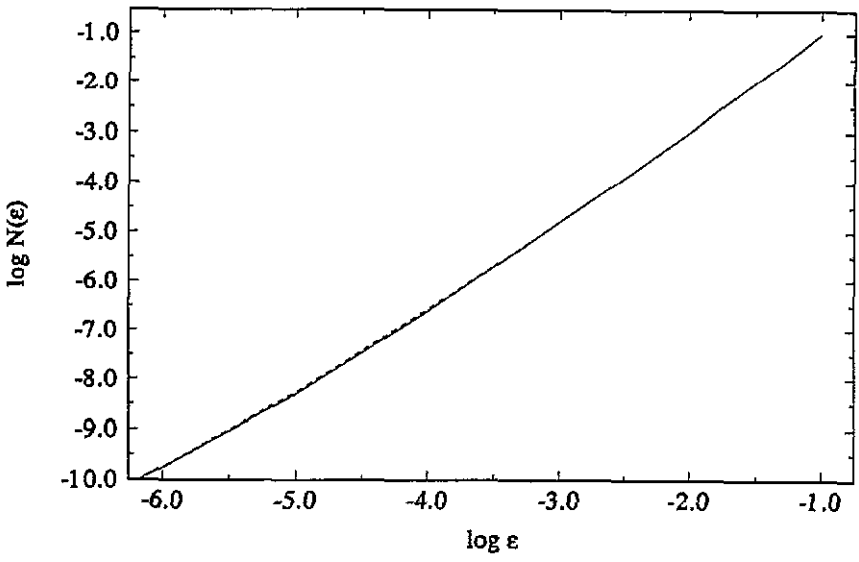

Figure 3. The figure illustrates equality of the capacity dimensions of the sets $(\nu_i, K(\nu_i))$ and $(\nu_i, K_a(\nu_i))$, where ν_i belong to the set of 18000 points of the form $\nu_i = i \times \gamma - [i \times \gamma]$ for which the exact critical function $K(\nu)$ is calculated with a sufficient accuracy. The dimensions are given by slopes of the two curves. The broken curve corresponds to $K(\nu_i)$ and the full curve to $K_a(\nu_i)$.

is obtained using (5), and the perturbation expansion of only tenth order which gave 32 values of the function $L_1(m/n)$ via (4). The curves $(\epsilon, \log N(\epsilon))$ for the two sets coincide within the accuracy, which shows that the scaling of $K(\nu)$ and $K_a(\nu)$ are numerically the same. We then proceeded to calculate the fractal dimensions of the graph $(\nu, K_a(\nu))$ by estimating a limit of fractal dimensions of subsets of the graph with successively larger number of points including those that are close to the rationals. The results are presented in figures 4 and 5.

For the standard map and the two-waves model we have no direct way of calculating the critical function at a number of points which is sufficient for estimating dimensions. An application of the Greene criterion to obtain the critical function at let us say 5000 points is practically impossible. So in these two cases we calculated the fractal dimensions of the approximate critical functions only, and obtained that they coincide, within the error, with the fractal dimensions for the semi-standard map. Finally we calculated the fractal dimensions of the exponent of the Brjuno function and obtained the same numbers within *the accuracy of the computations. We conclude that the spectra of calculated dimensions* are numerically the same. The spectra are illustrated in figure 5.

Let us now briefly describe the algorithm which we used for calculations of the fractal dimensions. We take a sequence of ϵ_n to be of the form $\epsilon_n = 2^{-n}$, and identify all points in a box of size ϵ_n^2 with one of the corners. The identification is achieved by chopping all digits after the nth in the binary expansion. After the identification the number of the same points gives the number of points in a box and the number of distinct points gives the number of boxes. In this way all the information which is needed for the calculation of dimensions is stored in a one-dimensional array. A similar algorithm has been proposed in [21].

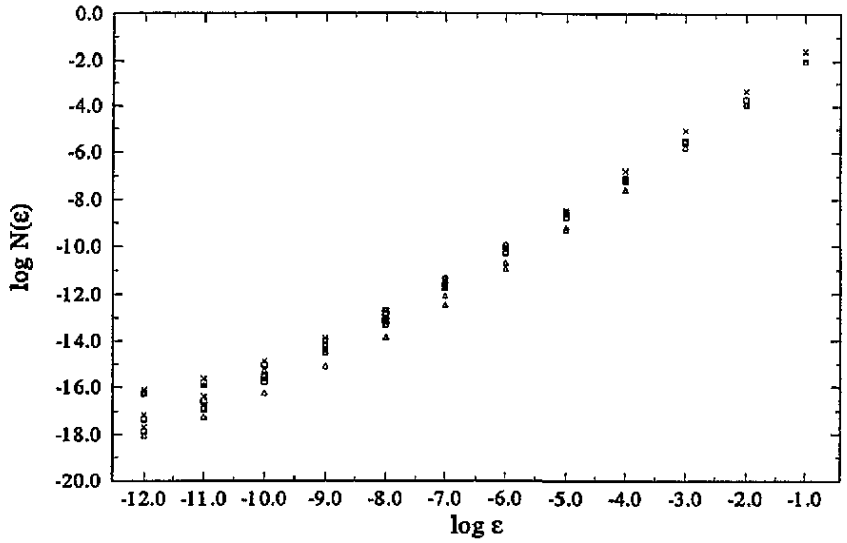

Figure 4. The figure gives representative points on the curves $(\log(\epsilon), \log(N(\epsilon)))$ for the sets $(\nu_i, K_a(\nu_i))$ where $\max i = 100\,000$, $300\,000$ and $600\,000$. The curves for the SSM (△), SM (○), TWM (×) and the exponent of the Brjuno function (□) are presented.

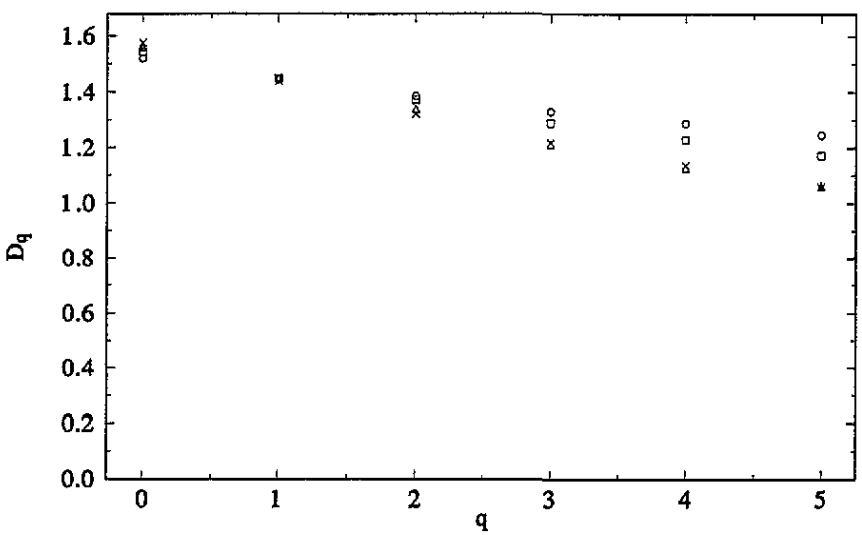

Figure 5. The first few fractal dimensions, corresponding to $q = 0, 1, 2, 3, 4, 5$ in the formula 10, of the critical functions for the SSM (△), SM (○), TWM (×) and the exponent of the Brjuno function (□).

4. Summary

Our results can be summarized as follows. First, the complexity, understood as the Renyi spectra, of the exact and the approximate critical functions, where $K_a(\nu)$ is obtained from a smooth approximation of the function $L_1(\nu)$, are practically indistinguishable. Second, the complexity of $K_a(\nu)$ does not depend on the form of the function $L_1(\nu)$, and is essentially

determined by the fact that $K(\nu)$ and its Gauss transform $K(\nu')$ form the continuous function $L_1(\nu)$. From these two results we conclude that the complexity of the critical functions for a class of Hamiltonian systems are practically indistinguishable. The complexity is determined by the arithmetic nature of the problem of small denominators and not by the particular form of the dynamics.

The results of numerical calculations presented in this paper need theoretical explanation. These results should also be extended to other systems with small denominators.

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